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# Convergence analysis of the parallel multisplitting TOR method

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## Abstract

In this paper, we propose the parallel multisplitting TOR method, for solving a large nonsingular systems of linear equations  $Ax = b$ . These new methods are a generalization and an improvement of the relaxed parallel multisplitting method (Formmer and Mager, 1989) and parallel multisplitting AOR Algorithm (Wang Deren, 1991). The convergence theorem of this new algorithm is established under the condition that the coefficient matrix  $A$  of linear systems is an H-matrix. Some results also yield new convergence theorem for TOR method.

**Keywords:** Multisplitting; Parallel Multisplitting TOR method; H-matrix; Convergence; Iterative method

**AMS classifications:** 65F10

## 1. Introduction

Suppose that we are given a linear system

$$Ax = b, \quad (1)$$

where  $A \in R^{n \times n}$  is a nonsingular matrix,  $x, b \in R^n$  are vectors. In order to compute the solution of (1) iteratively, O'Leary and White propose multisplitting methods in [6] which are based on several splittings of the matrix  $A$ . More precisely, in [6] a multisplitting of  $A$  is defined as a collection of triples  $(M_k, N_k, E_k)$ ,  $k = 1, 2, \dots, K$ , such that for all  $k$ ,  $M_k, N_k, E_k$  are  $n \times n$  matrices, each  $M_k$  is nonsingular,  $A = M_k - N_k$ , and  $E_k$  is a diagonal matrix with nonnegative entries satisfying  $\sum_{k=1}^K E_k = I$ . The corresponding multisplitting method to solve (1) is given by the iteration

$$x^{m+1} = \sum_{k=1}^K E_k y^{m,k}, \quad m = 0, 1, \dots, \quad (2)$$

where

$$M_k y^{m,k} = N_k x^m + b, \quad k = 1, 2, \dots, K$$

This multisplitting method has a natural parallelism, since the calculations of  $y^{m,k}$  for various  $k$  are independent and may therefore be performed in parallel. Moreover, the  $i$ -th component of  $y^{m,k}$  need not be computed if the corresponding diagonal entry of  $E_k$  is zero. This may result in considerable savings of computational time. Convergence results for method (2) were first given in [6]. Later, Neumann and Plemmons [5] obtained more qualitative results for one of the cases considered in [6].

In this paper, we present parallel multisplitting TOR<sup>1</sup> method (see Section 2) for solving large nonsingular system of linear equations (1) in Section 2. These new methods are a generalization and an improvement on the methods of [2, 9]. The convergence of this new method is discussed, under the condition that the coefficient matrix  $A$  is an  $H$ -matrix, in Section 3. We obtain the corresponding convergence theorems.

## 2. Parallel multisplitting TOR method

Suppose that  $A$  is a nonsingular  $n \times n$  matrix, for  $k = 1, 2, \dots, K$ ,  $L_k, F_k, U_k, E_k$  are  $n \times n$  matrices, and  $L_k$  and  $F_k$  are strictly lower triangular matrix satisfying

(1)  $A = D - L_k - F_k - U_k$ , where  $D = \text{diag}(A)$  are  $n \times n$  and are diagonal matrix and nonsingular, and  $U_k$  are strictly upper triangular matrix.

(2)  $\sum_{k=1}^K E_k = I$  ( $n \times n$ -identity matrix), where each  $E_k$  is diagonal matrix and  $E_k \geq 0$ .

Then the collection of triples  $(D - L_k - F_k, U_k, E_k)$  ( $k = 1, 2, \dots, K$ ) is called a multisplitting of  $A$ .

For real numbers  $\omega, \alpha$  and  $\beta$ , we define the function  $G_k : R^n \rightarrow R^n$  by

$$\begin{aligned} G_k(x) &= [D - (\alpha L_k + \beta F_k)]^{-1} \\ &\quad \times \{[(1 - \omega)D + (\omega - \alpha)L_k + (\omega - \beta)F_k + \omega U_k]x + \omega b\} \\ &\quad k = 1, 2, \dots, K. \end{aligned}$$

### Parallel Multisplitting TOR Method

For any starting vector  $x^0 \in R^n$ ,

$$x^{m+1} = \sum_{k=1}^{k=K} E_k G_k(x^m). \quad m = 0, 1, 2, \dots$$

until convergence, where

$$\begin{aligned} G_k(x^m) &= [D - (\alpha L_k + \beta F_k)]^{-1} \\ &\quad \times \{[(1 - \omega)D + (\omega - \alpha)L_k + (\omega - \beta)F_k + \omega U_k]x^m + \omega b\}. \end{aligned}$$

If we define the matrix

$$\begin{aligned} T_{\text{MTOR}}(\omega, \alpha, \beta) &= \sum_{k=1}^K E_k [D - \alpha L_k - \beta F_k]^{-1} \\ &\quad \times [(1 - \omega)D + (\omega - \alpha)L_k + (\omega - \beta)F_k + \omega U_k] \end{aligned} \quad (3)$$

<sup>1</sup> TOR method i.e. Two parameters Overrelaxation method, the strictly lower triangular part of matrix  $A$  is split into two parts,  $L$  and  $F$ . Also,  $L$  and  $F$  are multiplied deferent factors. Moreover see [4].

and the vector

$$g_{\text{MTOR}}(\omega, \alpha, \beta) = \sum_{k=1}^K E_k [D - \alpha L_k - \beta F_k]^{-1} (\omega b), \quad (4)$$

Then from Parallel Multisplitting TOR Method we get

$$x^{m+1} = T_{\text{MTOR}}(\omega, \alpha, \beta)x^m + g_{\text{MTOR}}(\omega, \alpha, \beta), \quad m = 0, 1, 2, \dots \quad (5)$$

It is easy to see that the iterative method (5) converges if and only if

$$\rho(T_{\text{MTOR}}(\omega, \alpha, \beta)) < 1,$$

see [8] or [12].

Obviously, if  $\alpha = \beta$ , then iterative method (5) will reduce to the parallel multisplitting AOR<sup>2</sup> algorithm [9]. If  $K=1$ , then method (5) will reduce to the TOR method [4]. We call method (5) a parallel multisplitting TOR method or  $T_{\text{MTOR}}(\omega, \alpha, \beta)$ -method. Furthermore, we observe that for specific values of parameters  $\alpha, \beta$  and  $\omega$ , the  $T_{\text{MTOR}}(\omega, \alpha, \beta)$ -method will reduce to the following methods:

$T_{\text{MTOR}}(1, 0, 0)$ -method =  $T_{\text{MGJ}}$ -method, is called the parallel multisplitting Jacobi method.

$T_{\text{MTOR}}(1, 1, 1)$ -method =  $T_{\text{MGS}}$ -method, is called the parallel multisplitting G-S method.

$T_{\text{MTOR}}(\omega, 0, 0)$ -method =  $T_{\text{MJOR}}$ -method, is called the parallel multisplitting JOR method.

$T_{\text{MTOR}}(\omega, \omega, \omega)$ -method =  $T_{\text{MSOR}}$ -method, is called the parallel multisplitting SOR method.

$T_{\text{MTOR}}(\omega, \alpha, \alpha)$ -method =  $T_{\text{MAOR}}$ -method, is called the parallel multisplitting AOR method.

Here, let  $T_{\text{MSOR}}$ -method be the relaxed parallel multisplitting method in [2]; and  $T_{\text{MAOR}}$ -method be the parallel multisplitting AOR algorithm in [9]. Thus,  $T_{\text{MTOR}}(\omega, \alpha, \beta)$ -method is an improvement and a generalization algorithm of [2, 9].

### 3. Convergence

First we need to introduce several known concepts and useful lemmas.

We say that a vector  $x \in R^n$  is nonnegative (positive), denoted  $x \geq 0$  ( $x > 0$ ), if  $x_i \geq 0$  ( $x_i > 0$ ) holds for all components of  $x = (x_1, x_2, \dots, x_n)^T$ .

Similarly, a matrix  $A$  is said to be nonnegative, if all its entries are nonnegative.

We compare two matrices  $A \geq B$ , when  $A - B \geq 0$ , and two vectors  $x \geq y$  ( $x > y$ ), when  $x - y \geq 0$  ( $x - y > 0$ ). Given a matrix  $A = (a_{ij})$  we define the absolute value of  $A$ ,  $|A| = (|a_{ij}|)$ ; it follows that  $|A| \geq 0$  and further  $|AB| \leq |A||B|$  for any two matrices  $A$  and  $B$ .

For any matrix  $A = (a_{ij})$ , if  $a_{ij} \leq 0$  for  $i \neq j$  and  $A^{-1} \geq 0$ , then  $A$  is called  $M$ -matrix (see [8]).

For any matrix  $A = (a_{ij}) \in R^{n \times n}$ , we define its comparison matrix  $\langle A \rangle = (\langle a_{ij} \rangle)$  by

$$\langle a_{ij} \rangle = \begin{cases} |a_{ij}| & \text{if } i = j, \\ -|a_{ij}| & \text{if } i \neq j. \end{cases}$$

A matrix  $A$  is called  $H$ -matrix if its comparison matrix  $\langle A \rangle$  is an  $M$ -matrix.

<sup>2</sup> AOR method i.e. Accelerated Overrelaxation Method, moreover see [3].

**Definition 1.** The splitting  $A = B - C$  is called

- (1) regular if  $B^{-1} \geq 0$  and  $C \geq 0$  [8];
- (2) weak regular if  $B^{-1} \geq 0$  and  $B^{-1}C \geq 0$  [1].

Now we introduce several useful lemmas.

**Lemma 1** (Formmer and Mager [2]). Let  $A$  be an  $H$ -matrix,  $D = \text{diag}(A)$  and  $A = D - B$ ; then

- (1)  $A$  is nonsingular;
- (2)  $|A^{-1}| \leq \langle A \rangle^{-1}$ ;
- (3)  $|D|$  is nonsingular and  $\rho(|D|^{-1}|B|) < 1$ .

**Lemma 2** (Varga [8]). Suppose  $A, B$  are such that  $|A| \leq B$ ; then  $\rho(A) \leq \rho(B)$ .

**Lemma 3** (Varga [8]). Suppose  $A$  be a nonnegative irreducible matrix; then the spectral radius  $\rho(A)$  of  $A$  is an eigenvalue of  $A$  and the eigenvector  $x$  corresponding to  $\rho(A)$  such that  $x > 0$ .

**Theorem 1.** Suppose  $A$  is an  $H$ -matrix, with a multisplitting

$$(D - L_k - F_k, U_k, E_k), \quad k = 1, 2, \dots, K,$$

such that

$$\langle A \rangle = |D| - |L_k| - |F_k| - U_k = |D| - |B|.$$

Then  $T_{\text{MTOR}}(\omega, \alpha, \beta)$ -method (5) converges provided the parameters  $\omega, \alpha, \beta$  satisfy

$$0 \leq \alpha \leq \omega, \quad 0 < \omega < \frac{2}{1 + \rho(|D|^{-1}|B|)}, \quad (6)$$

$$0 \leq \beta \leq \omega, \quad 0 < \omega < \frac{2}{1 + \rho(|D|^{-1}|B|)}. \quad (7)$$

**Proof.** We will show that  $\rho(|T_{\text{MTOR}}(\omega, \alpha, \beta)|) < 1$ , where  $T_{\text{MTOR}}(\omega, \alpha, \beta)$  is the matrix given by (3) and (4). Since  $\rho(T_{\text{MTOR}}(\omega, \alpha, \beta)) \leq \rho(|T_{\text{MTOR}}(\omega, \alpha, \beta)|)$  by Lemma 2, the theorem is then proved.  $\square$

Since  $A$  is an  $H$ -matrix,  $D$  is a diagonal matrix,  $L_k$  and  $F_k$  are strictly lower triangular matrix, we easily see that  $D - \alpha L_k - \beta F_k$  are  $H$ -matrix for  $k = 1, 2, \dots, K$ . Using Lemma 1(2) and the definition of comparison matrix, we get

$$|(D - \alpha L_k - \beta F_k)^{-1}| \leq \langle D - \alpha L_k - \beta F_k \rangle^{-1} = (|D| - \alpha |L_k| - \beta |F_k|)^{-1}$$

(1) First let the inequality  $0 \leq \alpha \leq \omega, 0 \leq \beta \leq \omega, 0 < \omega \leq 1$  hold. For  $k = 1, 2, \dots, K$ , we define the matrices

$$M_k = |D| - \alpha |L_k| - \beta |F_k|, \quad (8)$$

$$N_k^1 = (1 - \omega)|D| + (\omega - \alpha)|L_k| + (\omega - \beta)|F_k| + \omega|U_k|. \quad (9)$$

From (8) and (9),

$$N_k^1 = M_k - \omega|D| - \omega|B| = M_k - \omega(|D| - |B|). \quad (10)$$

Taking the absolute values on either side of (3) and (4) we have

$$|T_{\text{MTOR}}(\omega, \alpha, \beta)| \leq \sum_{k=1}^K E_k M_k^{-1} N_k^1 \quad (11)$$

$$\leq \sum_{k=1}^K E_k M_k^{-1} [M_k - \omega(|D| - |B|)]. \quad (12)$$

$$\leq I - \omega \sum_{k=1}^K E_k M_k^{-1} |D| (I - |D|^{-1} |B|) \quad (13)$$

Let  $e = [1, 1, \dots, 1]^T \in R^n$ . Since  $|D|^{-1}|B|$  is nonnegative, the matrix  $J_\epsilon = |D|^{-1}|B| + \epsilon ee^T$  has only positive entries and is irreducible for any  $\epsilon > 0$ . By Lemma 3, we know that  $\rho(J_\epsilon)$  is an eigenvalue of  $J_\epsilon$  and the corresponding eigenvector  $x_\epsilon \geq 0$  is such that

$$J_\epsilon x_\epsilon = (|D|^{-1}|B| + \epsilon ee^T)x_\epsilon = \rho(J_\epsilon)x_\epsilon.$$

Moreover, since  $0 < \omega \leq 1$  we have

$$1 - \omega + \omega\rho(|D|^{-1}|B|) < 1.$$

By continuity of the spectral radius we also get

$$1 - \omega + \omega\rho(J_\epsilon) < 1 \quad (14)$$

if  $\epsilon > 0$  is sufficiently small.

From (11)–(13) we have

$$|T_{\text{MTOR}}(\omega, \alpha, \beta)| \leq I - \omega \sum_{k=1}^K E_k M_k^{-1} |D| [I - (|D|^{-1}|B| + \epsilon ee^T)] \quad (15)$$

$$= I - \omega \sum_{k=1}^K E_k M_k^{-1} |D| (I - J_\epsilon) \quad (16)$$

and multiplying by  $x_\epsilon$ ,

$$|T_{\text{MTOR}}(\omega, \alpha, \beta)|x_\epsilon \leq x_\epsilon - \omega \sum_{k=1}^K E_k M_k^{-1} |D| (1 - \rho(J_\epsilon))x_\epsilon. \quad (17)$$

From the definition of  $M_k$ ,  $M_k$  are  $H$ -matrices, by Lemma 1(2), we get

$$M_k \leq |D|, \quad M_k^{-1} \geq |D|^{-1}.$$

By (17) and (14) we have

$$|T_{\text{MTOR}}(\omega, \alpha, \beta)|x_\epsilon \leq x_\epsilon - \omega \sum_{k=1}^K E_k |D|^{-1} |D| (I - \rho(J_\epsilon))x_\epsilon \quad (18)$$

$$= (1 - \omega + \omega \rho(J_\epsilon))x_\epsilon \quad (19)$$

$$< x_\epsilon. \quad (20)$$

By [8, Exercise 2, p. 48], the  $\rho(|T_{\text{MTOR}}(\omega, \alpha, \beta)|) < 1$  holds.

(2) Next let the inequality  $1 < \alpha \leq \omega, 1 < \beta \leq \omega, 1 < \omega < 2/(1 + \rho(|D|^{-1}|B|))$  hold. We define matrices

$$N_k^2 = (\omega - 1)|D| + (\omega - \alpha)|L_k| + (\omega - \beta)|F_k| + \omega|U_k|. \quad (21)$$

From (8) and (21), then

$$N_k^2 = M_k - [(2 - \omega)|D| - \omega|B|] \quad (22)$$

Taking the absolute values on either side of (3) and (4) we have

$$|T_{\text{MTOR}}(\omega, \alpha, \beta)| \leq \sum_{k=1}^K E_k M_k^{-1} N_k^2 \quad (23)$$

$$\leq \sum_{k=1}^K E_k M_k^{-1} [M_k - ((2 - \omega)|D| - \omega|B|)] \quad (24)$$

$$\leq I - \sum_{k=1}^K E_k M_k^{-1} |D| [(2 - \omega)I - \omega|D|^{-1}|B|]. \quad (25)$$

As before, let  $e = [1, 1, \dots, 1]^T \in R^n$  and let  $x_\epsilon > 0$  denote a vector satisfying  $J_\epsilon = (J + \epsilon ee^T)x_\epsilon = \rho(J_\epsilon)x_\epsilon$ , where  $\epsilon > 0$  is sufficiently small such that  $\omega - 1 + \omega \rho(J_\epsilon) < 1$ , since  $1 < \omega < 2/(1 + \rho(|D|^{-1}|B|))$ .

From (23)–(25) we get

$$|T_{\text{MTOR}}(\omega, \alpha, \beta)| \leq I - \sum_{k=1}^K E_k M_k^{-1} |D| [(2 - \omega)I - \omega J_\epsilon]. \quad (26)$$

Then by multiplying by  $x_\epsilon$ , we have

$$|T_{\text{MTOR}}(\omega, \alpha, \beta)|x_\epsilon \leq x_\epsilon - \sum_{k=1}^K E_k |D|^{-1} |D| [2 - \omega - \omega \rho(J_\epsilon)]x_\epsilon \quad (27)$$

$$= x_\epsilon - [2 - \omega - \omega \rho(J_\epsilon)]x_\epsilon \quad (28)$$

$$= [\omega - 1 + \omega \rho(J_\epsilon)]x_\epsilon \quad (29)$$

$$< x_\epsilon. \quad (30)$$

Thus,  $\rho(|T_{\text{MTOR}}(\omega, \alpha, \beta)|) < 1$  follows again by [8, exercise 2, p. 48]. By the assumption of the theorem, this completes the proof of Theorem 1.  $\square$

The condition that  $A$  is an  $H$ -matrix covers several interesting cases. We consider some of them in the next corollary.

**Corollary 1.** *Let  $A$  be an  $n \times n$  matrix, such that one of the following conditions*

- (1)  *$A$  is an  $M$ -matrix;*
- (2)  *$A$  is strictly or irreducibly diagonally dominant matrix;*
- (3)  *$A$  is symmetric positive-definite  $L$ -matrix, holds, and*

$$(D - L_k - F_k, U_k, E_k), \quad k = 1, 2, \dots, K$$

*be a multisplitting of  $A$ . If*

$$\langle A \rangle = |D| - |L_k| - |F_k| - |U_k| = |D| - |B|$$

*then the  $T_{\text{MTOR}}(\omega, \alpha, \beta)$ -method converges for any starting vector  $x^0 \in R^n$  provided the parameters  $\omega, \alpha, \beta$  satisfy*

$$0 \leq \alpha, \quad \beta \leq \omega, \quad 0 < \omega < \frac{2}{1 + \rho(|D|^{-1}B)}.$$

We remark that for the multisplitting case  $K=1$ , this is Parallel Multisplitting TOR Method reduce to the standard method. Thus our results are generalization the most results of [2], [3], [4], [5], [8], [9], [12].

**Example 1.** Let

$$A = \begin{bmatrix} 1 & 0 & -\theta_1 \\ 0 & 1 & \theta_2 \\ -\theta_2 & \theta_1 & 1 \end{bmatrix} = D - B,$$

where  $0 < \theta_1$  and  $0 < \theta_2$  are real parameters, and satisfy  $|\theta_1| + |\theta_2| < 1$ , with

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & \theta_1 \\ 0 & 0 & -\theta_2 \\ \theta_2 & -\theta_1 & 1 \end{bmatrix}.$$

It is easy to verify that  $\rho(|D|^{-1}|B|) = \sqrt{2\theta_1\theta_2}$ , from Theorem 1 or Corollary 1, the  $T_{\text{MTOR}}(\omega, \alpha, \beta)$ -method converges for

$$0 \leq \alpha < \omega, \quad 0 < \omega < 2/(1 + \sqrt{2\theta_1\theta_2}),$$

$$0 \leq \beta < \omega, \quad 0 < \omega < 2/(1 + \sqrt{2\theta_1\theta_2}).$$

When the parameters  $\theta_1 = \frac{1}{2}$  and  $\theta_2 = \frac{1}{3}$ , then the  $T_{\text{MTOR}}(\omega, \alpha, \beta)$ -method converges for

$$0 \leq \alpha < \omega, \quad 0 < \omega < 3 - \sqrt{3},$$

$$0 \leq \beta < \omega, \quad 0 < \omega < 3 - \sqrt{3}.$$

**Example 2.** Let

$$A = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & -v \\ -v & 1 & 0 & \cdots & 0 & 0 \\ 0 & -v & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & -v & 1 \end{bmatrix} = D - B,$$

where  $v$  is a real parameter and satisfies  $|v| < 1$ , with

$$D = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & v \\ v & 0 & 0 & \cdots & 0 & 0 \\ 0 & v & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & v & 0 \end{bmatrix}.$$

It is easy to verify that  $\rho(|D|^{-1}|B|) = v$ , from Theorem 1 or Corollary 1, the  $T_{\text{MTOR}}(\omega, \alpha, \beta)$ -method converges for

$$0 \leq \alpha < \omega, \quad 0 < \omega < 2/(1+v)$$

$$0 \leq \beta < \omega, \quad 0 < \omega < 2/(1+v)$$

When the parameters  $v = \frac{1}{3}$ , then the  $T_{\text{MTOR}}(\omega, \alpha, \beta)$ -method converges for

$$0 \leq \alpha < \omega, \quad 0 < \omega < \frac{5}{3}$$

$$0 \leq \beta < \omega, \quad 0 < \omega < \frac{5}{3}$$

#### 4. Remarks

(1) Theorem 1 was proved under the condition that  $A$  is an  $H$ -matrix. It is worth studying that Theorem 1 still holds under the condition that  $A$  is a symmetric positive-definite matrix,  $L$ -matrix, etc. Moreover, the problem of the optimal choice of three parameters is also interesting.

(2) In order to adapt better MIMD systems, the chaotic variation of parallel multisplitting TOR method is worth studying.

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